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NOTE ON INFINITE GRAPHS

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Some relations between the number of nodes and edges and the degrees of the nodes in infinite graphs are obtained. The structure of infinite connected graphs which have no $2-\infty$ trails is investigated with the help of these. It is shown, for example, that any such graph G has $|G|$ nodes of odd degree.

1. Introduction and terminology

In this paper the term *graph* always means an undirected graph without loops, which may be infinite and may contain multiple edges. The Axiom of Choice will be assumed. If S is a set, then $|S|$ will denote the cardinal of S. If G is a graph, then the set of all nodes of G will be denoted by $N(G)$, the set of all edges of G by $E(G)$, and $|N(G) \cup E(G)|$ by $|G|$, the graph G will be called *infinite* if $|G| \geqslant \aleph_0$. If n is a node of the graph G, then the cardinal of the set of all edges of G incident with n (which may be finite or infinite) will be called the *degree* of n in G and denoted by $d_G(n)$. $D(G)$ will denote the set of all degrees occurring in G, $\Delta(G)$ the least upper bound of $D(G)$, and for each finite or infinite cardinal x the set of all degrees $\geqslant x$ occurring in G will be denoted by $D_{\geqslant x}(G)$. The set of all bridges of G will be denoted by $E_{\mathrm{br}}(G)$, $E(G) - E_{\mathrm{br}}(G)$ by $E_{\mathrm{bl}}(G)$, the set of all those edges in $E_{\mathrm{br}}(G)$ and in $E_{\mathrm{bl}}(G)$ which are incident with the node n of G by $E_{\mathrm{br}}(n, G)$ and $E_{\mathrm{bl}}(n, G)$, respectively. $E(n_1, n_2; G)$, where n_1 and n_2 are any two distinct nodes of G, will denote the set of all edges of G which join n_1 and n_2. Any set of graphs will be called *mutually edge-disjoint*, or m.e.d. for short, if no two of them have an edge in common. A *circuit* is any finite connected graph in which all nodes have degree 2. The terminology of [4] will be used except where it differs from the above, and for the definitions of *way*, *infinite way*, *infinite trail*, see [2]. H and K being two non-empty subgraphs of a graph, a way one of whose end-nodes belongs to H and one to K, and which has no other node in common with $H \cup K$, will be called an *(H,K)-way*. Here H and K may consist of isolated nodes.

This paper will establish some general properties of infinite graphs, and will use these to investigate the structure of those infinite connected graphs which do not have infinite trails. It was noted in another context in [2] that every such graph contains at least two nodes having odd degree.

2. Some general properties of infinite graphs

The following is known and easy to verify:

(1) *For any graph G with $E(G)$ infinite,*

$$|E(G)| = \sum_{n \in N(G)} d_G(n).$$

The following is a slight generalisation of a famous result of König [5, Ch. VI, Theorem 3].

Theorem 2.1. *Any infinite graph G either has $|G|$ connected components, or it contains at least one node having infinite degree, or it contains a $1-\infty$ way.*

Proof. If G has fewer than $|G|$ connected components, then at least one of them is infinite, and any infinite connected graph either contains a node having infinite degree or an infinite way by König's theorem.

Corollary 2.2. *Let T be a $1-\infty$ trail of a graph and let $G(T)$ denote the graph consisting of the nodes and edges of T. Then either at least one node occurs infinitely often in T, or $G(T)$ contains a $1-\infty$ way.*

Definition 2.3. Let W be a $1-\infty$ way and n a node. The union of W and any infinite set of different (n, W)-ways, any two of which have just n in common and none of which contains an edge of W, will be called a *fan* with *base* W and *hub* n. Halin [3] proved that every 2-connected graph which contains a $1-\infty$ way but no $2-\infty$ way contains a fan. Halin's theorem will be used later, a generalization of it will be given here.

Theorem 2.4. (A) *Let G be any connected graph which contains a fan and which does not contain a $2-\infty$ way. Then if F is any fan of G and W is any $1-\infty$ way of G, $F \cup W$ contains a fan with the same hub as F and with base W.*

(B) *Let G be any graph which contains a $1-\infty$ way but no $2-\infty$ way, which is 2-connected, and which remains 2-connected when any set of at most $k-2$ nodes of infinite degree are deleted from it, where k is an integer $\geqslant 2$ (this is the case when G is k-connected). Then if W is any $1-\infty$ way of G, there exist $k-1$ fans in G with $k-1$ different hubs, each of which has base W.*

Proof of (A). Let n denote the hub and W' the base of F. If $W \subseteq W'$ or $W \supseteq W'$, then the assertion is obviously true. Otherwise let $w_1, w_2, w_3, \ldots$ denote the nodes of W' in order starting with its end-node, let $W_1, W_2, W_3, \ldots$ denote the different (n, W')-ways of F, and let $w_{i_1}, w_{i_2}, w_{i_3}, \ldots$ denote their end-nodes on W', the notation being chosen so that $i_1 < i_2 < i_3 < \ldots$. For $j = 1, 2, 3, \ldots$, let W'_j denote the way consisting of W_j together with the part of W' which joins w_{i_j} and $w_{i_{j+1}-1}$. Then it is easy to see that $W'_1, W'_2, W'_3, \ldots$ are distinct ways, any two of them have just n in common, and their union includes all of w_g with $g \geqslant i_1$. Since G contains no $2-\infty$ way, W and W' have infinitely many nodes in common, therefore infinitely many of $W'_1, W'_2, W'_3, \ldots$ have a node $\neq n$ in common with W; let $W'_{h_1}, W'_{h_2}, W'_{h_3}, \ldots$ denote the subsequences of these, and for $i = 1, 2, 3, \ldots$ let n_{h_i} denote the first node $\neq n$ belonging to W encountered when W'_{h_i} is followed starting with n, finally let W''_{h_i} denote that part of W'_{h_i} which joins n and n_{h_i}. Then $W \cup W''_{h_1} \cup W''_{h_2} \cup W''_{h_3} \cup \ldots$ is a fan with base W and hub n and is contained in $F \cup W$. (A) is now proved.

Proof of (B). By Halin's theorem, G contains a fan, let n_1 denote its hub. By (A), G contains a fan with hub n_1 and base W. So (B) is true if $k = 2$. Suppose that $k > 2$. By hypothesis, $G - n_1$ is 2-connected, obviously it contains a $1-\infty$ way W^1 contained in W and no $2-\infty$ way, therefore by Halin's theorem it contains a fan, let n_2 denote its hub. By (A), G contains a fan with hub n_2 and base W^1. So (B) is true if $k = 3$. Repeating this argument $k-1$ times, (B) is proved for each integer $k \geqslant 2$.

A result similar to Theorem 2.1 is the following.

Theorem 2.5. *If G is a graph and $|G| > g \geqslant \aleph_0$, then either G has more than g connected components, or at least one node has degree $> g$ in G.*

Proof. Proof is by reductio ad absurdum. Suppose on the contrary that G has at most g connected components and every node of G has degree $\leqslant g$. Then G has a connected component G' such that $|G'| > g$, for otherwise $|G| \leqslant g^2 = g$.

Let n_0 be a node of G' and let D_i denote the set of all nodes of G' whose distance from n_0 is i, for $i = 1, 2, 3, \ldots$. Clearly, $N(G') = \{n_0\} \cup D_1 \cup D_2 \cup D_3 \ldots$ and $|D_i| \leqslant g$ for $i = 1, 2, 3, \ldots$. Therefore $|N(G')| \leqslant g$. It follows from this and (1) that $|E(G')| \leqslant g$. Consequently, $|G'| \leqslant g$. But it was shown above that $|G'| > g$. This contradiction proves Theorem 2.5.

Theorem 2.6. (A) *For any graph G, $|E(G)| \geqslant \Delta(G)$.*

(B) *For any infinite graph G exactly one of the following three alternatives holds:*

(I) *G has more than $|E(G)|$ isolated nodes and $|E(G)| < |N(G)| = |G|$;*

(II) $|E(G)| = |N(G)| = |G|$;

(III) $|E(G)| = \Delta(G) = |G| > |N(G)|$, *$|E(G)|$ is the least upper bound of $\{|E(n_1, n_2; G)| : n_1, n_2 \in N(G)\}$, and either one or more nodes have degree $|E(G)|$ in G, or else $N(G)$ is infinite, G contains no node having maximum degree, and $|E(G)|$ is a limit cardinal $> \aleph_0$.*

(C) *For any infinite graph G for which there exists a cardinal $p < |E(G)|$ such that each pair of nodes with at most a finite number of exceptions are joined by at most p edges, and each pair of nodes are joined by fewer than $|E(G)|$ edges, and which does not contain more than $|E(G)|$ isolated nodes, $|E(G)| = |N(G)| = |G|$.*

(D) *For each infinite graph G in which each pair of nodes are joined by at most $|N(G)|$ edges and which does not contain more than $|E(G)|$ isolated nodes, $|E(G)| = |N(G)| = |G|$.*

Proof of (A). Suppose on the contrary (reductio ad absurdum) that $|E(G)| < \Delta(G)$. It follows from the definition of $\Delta(G)$ that G contains a node having degree $> |E(G)|$, which is absurd.

Proof of (B). If G has more than $|E(G)|$ isolated nodes, then $|E(G)| < |N(G)|$ and therefore $|N(G)| = |G|$, so (I) holds.

In what follows suppose that G has at most $|E(G)|$ isolated nodes. In this case $E(G)$ is infinite, therefore, by (1), $|E(G)| \geqslant N(G)$, hence $|G| = |E(G)| \geqslant |N(G)|$. Therefore either $|E(G)| = |N(G)| = |G|$, in which case (II) holds, or else $|N(G)| < |E(G)| = |G|$. In what follows suppose that $|N(G)| < |E(G)| = |G|$.

By (1), $|E(G)| \leqslant \Delta(G)|N(G)|$ whenever $E(G)$ is infinite. Since $|N(G)| < |E(G)|$ by hypothesis and $\Delta(G) \leqslant |E(G)|$ by (A), we have that $|G| = |E(G)| = \Delta(G) > |N(G)|$.

Let m denote the least upper bound of $\{|E(n_1, n_2; G)|: n_1, n_2 \in N(G)\}$. If $N(G)$ is finite, then it is easy to see that $m = |E(G)|$. Suppose that $N(G)$ is infinite. Obviously, $m \leqslant \Delta(G)$. Each node of G has degree $\leqslant m|N(G)|$, therefore, by (1), $|E(G)| \leqslant m|N(G)|^2 = m|N(G)|$. Hence $|E(G)| \leqslant m$ since $|E(G)| > |N(G)|$. From $m \leqslant \Delta(G)$, $\Delta(G) = |E(G)|$ and $|E(G)| \leqslant m$ we have that $m = |E(G)|$.

One possibility is that one or more of the nodes of G have degree $\Delta(G) = |E(G)|$ in G, then (III) holds. If this is not the case, then G has no node of maximum degree. In this event, of course, $|N(G)| \geqslant \aleph_0$ and $\Delta(G) = |E(G)|$ is a limit cardinal, moreover $|E(G)| > |N(G)|$. Hence (III) holds again.

Obviously, (I), (II) and (III) are mutually exclusive. So (B) is now proved.

Proof of (C). By (B), either (II) or (III) holds for G, so $|N(G)| \leqslant |E(G)| = |G|$. To prove that $|N(G)| = |E(G)|$ suppose on the contrary that $|N(G)| < |E(G)|$ (reductio ad absurdum). If $N(G)$ is finite, then $|E(G)|$ is finite and then $|E(G)|$ is the sum of a finite number of cardinals each $< |E(G)|$, which is absurd, therefore $N(G)$ is infinite. Then if each pair of nodes is joined by at most p edges, we have $|E(G)| \leqslant p|N(G)|$, and otherwise $|E(G)| \leqslant p|N(G)| + e_1 + ... + e_q$, where q is a natural number and $e_1, ..., e_q < |E(G)|$. But this is absurd because p, $|N(G)|$, $e_1, ..., e_q < |E(G)|$. Therefore, $|E(G)| = |N(G)|$. (C) is now proved.

Proof of (D). $N(G)$ is infinite because otherwise G would be finite. Therefore, $|E(G)| \leqslant |N(G)|$. By (B), $|E(G)| \geqslant |N(G)|$. Hence $|E(G)| = |N(G)| = |G|$. Theorem 2.6 is now proved.

Theorem 2.7. (A) *If G is a graph and $\Delta(G)$ is infinite, then*

$$\Delta(G) = \sum_{y \in D_{\geqslant x}(G)} y$$

for every cardinal x such that $1 \leq x < \Delta(G)$, and also for $x = \Delta(G)$ *if G contains a node having maximum degree.*

(B) *If the graph G has at most $\Delta(G)$ connected components and $\Delta(G)$ is infinite, then $|E(G)| = \Delta(G) = |G|$.*

Proof of (A). For the cardinals x described, $D_{\geq x}(G) \neq \emptyset$ and $\Delta(G)$ is the least upper bound of $D_{\geq x}(G)$. It is easy to see that if a set of cardinals has an infinite least upper bound, then the sum of all the cardinals in the set is equal to the least upper bound. This proves (A).

Proof of (B). $|G| = \Delta(G)$, for otherwise $|G| > \Delta(G)$ by Theorem 2.5(A), so by Theorem 2.4, G contains a node having degree $> \Delta(G)$, which is contrary to the definition of $\Delta(G)$. From $|G| = \Delta(G)$ and Theorem 2.6(A) we have (B).

Theorem 2.8. *If G is a graph and n is any node of G such that $|E_{\mathrm{bl}}(n,G)| \geq \aleph_0$, then each relatively maximal set of m.e.d. circuits of G all containing n contains just $|E_{\mathrm{bl}}(n,G)|$ circuits.*

Proof. Proof is by reductio ad absurdum. Put $|E_{\mathrm{bl}}(n,G)| = f$. Suppose that C is a relatively maximal set of m.e.d. circuits of G, all containing n, with $|C| \neq f$. Then $|C| < f$ because every edge incident with n of every circuit of C belongs to $E_{\mathrm{bl}}(n,G)$. Let $E(C)$ denote the set consisting of all edges of all circuits of C. Let G' denote the union of all blocks of G with more than one edge to which n belongs. Then $n \in G' \subseteq G$, each circuit of G which contains n is contained in G', the set of edges of G' incident with n is $E_{\mathrm{bl}}(n,G)$, and G' is a connected graph.

Let the connected component of $G' - E(C)$ to which n belongs be denoted by G''. Every edge of G'' incident with n is a bridge of G'' because G'' contains no circuit containing n from the definition of C. Also $d_{G''}(n) = f$ because $|C| < f$. For each edge e of G'' incident with n let $G''(e)$ denote that connected component of $G'' - e$ which does not contain n; $G''(e_1) \cap G''(e_2) = \emptyset$ if $e_1 \neq e_2$ because otherwise G'' would contain a circuit containing both e_1 and e_2 and therefore e_1 and e_2 would not be bridges of G''. Since G' has no bridge, it follows that at least one edge of $E(C)$ is incident with some node of $G''(e)$ for each edge e of G'' incident with n. But this is impossible because $d_{G''}(n) = f$, whereas $|E(C)| < f$ because $|C| < f$ by hypothesis. Theorem 2.8 is now proved.

Remark 2.9. For each node of any graph there exists at least one relatively maximal set of m.e.d. circuits all containing the node (see [1, Lemma 1]).

Remark 2.10. It is possible that $E_{\mathrm{br}}(n,C) = \emptyset$ and yet $|E_{\mathrm{bl}}(n,G) - E(C)| = d_G(n) \geqslant \aleph_0$. This is the case, for example, whenever $d_G(n) \geqslant \aleph_0$ and in each block of G containing n the number of edges incident with n is odd and > 1.

3. Infinite graphs which have no $1-\infty$ or no $2-\infty$ trails

Definition 3.1. Let G be a graph. The connected components of the graph obtained by deleting from G all its bridges will be called the *clusters* of G. (If G has no bridge, then G is the only cluster of G.)

Remark 3.2. Each cluster of G is a single node incident only with bridges of G, or an isolated node of G, or the union of blocks of G each containing more than one edge, and each block of G containing more than one edge is a block of a cluster of G. Any edge of G is a bridge of G if and only if its end-nodes belong to different clusters of G. Two different clusters are joined by at most one edge.

Definition 3.3. Let G be a graph. To each cluster C of G assign a node $n(C)$, where $n(C) \notin G$ and $n(C) \neq n(C')$ whenever C and C' are different clusters of G. Join $n(C)$ and $n(C')$ by an edge if and only if there exists an edge of G joining the clusters C and C'. It may easily be verified that the graph so obtained contains no circuit. It will be called the *cluster-bridge tree*, or *cluster-bridge forest*, of G.

Remark 3.4. The graph defined above is connected if and only if G is connected.

Theorem 3.5. *Let G be any graph. The following three statements are equivalent:*

(1) *G has no $1-\infty$ trail.*

(2) *Every cluster of G is finite and G contains no $1-\infty$ way.*

(3) *Every cluster of G is finite and the cluster-bridge tree or forest of G contains no $1-\infty$ way.*

The following three statements are equivalent:

(4) *G has no $2-\infty$ trail.*

(5) *Every cluster of G is finite and G contains no $2-\infty$ way.*

(6) *Every cluster of G is finite and the cluster-bridge tree or forest of G contains no $2-\infty$ way.*

Proof. It will be shown that (4) implies that every cluster of G is finite. We first prove that every block of G is finite. Suppose on the contrary (reductio ad absurdum) that B is an infinite block of G. By Theorem 2.8, $E_{bl}(n,G)$ is finite for each node n of G, hence each node has finite degree in B. Therefore B contains a $1-\infty$ way by Theorem 2.1. Since G contains no $2-\infty$ way, it follows from this and Halin's theorem that B contains a fan. The hub of such a fan has infinite degree in B, whereas it was shown above that each node has finite degree in B. This contradiction proves that every block of G is finite.

We next prove that every cluster of G contains only a finite number of blocks. Suppose on the contrary (reductio ad absurdum) that C is a cluster of G which contains infinitely many blocks. Let H denote the block-cutnode graph of C. By Remark 3.2, each block of C contains more than one edge. Therefore each cutnode of C is incident with only a finite number of blocks of C, since by Theorem 2.8, $E_{bl}(n,G)$ is finite for each node n of G. It was shown above that each block of G is finite. It now follows that each node of H has finite degree. H is an infinite tree, so by Theorem 2.1, H contains a $1-\infty$ way. It follows that the notation can be chosen so that H contains a $1-\infty$ way whose nodes in order, starting from the end-node, are $b'_1, c'_1, b'_2, c'_2, \ldots$, where for $i = 1, 2, 3, \ldots$, b'_i corresponds to the block B_i of C and c'_i corresponds to the cutnode c_i of C which belongs to B_i and to B_{i+1}. Since for $i = 2, 3, 4, \ldots$, B_i is a block with more than one edge, and since c_{i-1} and c_i are distinct nodes of B_i, there exist two ways W_{i1} and W_{i2} in B_i, both of which have c_{i-1} and c_i as end-nodes, and have nothing except these two nodes in common. It is easy to see that $W_{11} \cup W_{21} \cup W_{31} \cup \ldots$ is a $1-\infty$ way, also $W_{12} \cup W_{22} \cup W_{32} \cup \ldots$ is a $1-\infty$ way, and these two $1-\infty$ ways have just the nodes $c_1, c_2, c_3, \ldots$ in common. Therefore the union of these two $1-\infty$ ways has a $2-\infty$ trail. This contradiction proves that every cluster of G contains only a finite number of blocks. It has been proved that every block of G is finite. Therefore (4) implies that every cluster of G is finite.

It follows at once that (4) implies (5).

(5) implies (6) for if the cluster-bridge tree or forest of G contains a $2-\infty$ way, then so does G since each cluster is a connected subgraph of G.

(6) implies (4). For suppose on the contrary (reductio ad absurdum) that (6) holds and G has a $2-\infty$ trail. Then the cluster-bridge tree or forest of G has a $2-\infty$ trail, because every cluster of G is finite. But the nodes and edges of any $2-\infty$ trail of a tree clearly constitute a $2-\infty$ way

contained in the tree, hence the cluster-bridge tree or forest of G contains a $2-\infty$ way, which is contrary to (6). This contradiction proves that (6) implies (4).

It has now been proved that (4), (5) and (6) are equivalent.

A fortiori (1) implies (2).

(2) implies (3) and (3) implies (1) by the same reasoning as was used to show that (5) implies (6) and (6) implies (1), with "$2-\infty$" replaced by "$1-\infty$" throughout. Theorem 3.5 is now proved.

Remark 3.6. (3) and (6) of Theorem 3.5 show very clearly how all graphs without $1-\infty$ trails and all graphs without $2-\infty$ trails are generated from finite connected graphs without bridges and from trees without $1-\infty$ ways and without $2-\infty$ ways, respectively.

Corollary 3.7. *Let G be a graph. The following three statements are equivalent:*

(1) *G has a $1-\infty$ trail but no $2-\infty$ trail.*

(2) *Every cluster of G is finite and G contains a $1-\infty$ way but no $2-\infty$ way.*

(3) *Every cluster of G is finite and the cluster-bridge tree or forest of G contains a $1-\infty$ way but no $2-\infty$ way.*

Definition 3.8. A cluster of a graph which is incident with exactly one bridge of the graph will be called an *end-cluster.*

Theorem 3.9. *If G is an infinite connected graph and has no $2-\infty$ trail, then all the clusters of G are finite and* either *at least one node has infinite degree in G and*

$$|E_{\mathrm{br}}(G)| = |C^*(G)| = |N^*(G)| = \Delta(G) = |G|,$$

where $C^*(G)$ is the set of all end-clusters of G and $N^*(G)$ is the set of all nodes having odd degree in G, or *G and its cluster-bridge tree are locally finite and contain a $1-\infty$ way but no $2-\infty$ way, $|E_{\mathrm{br}}(G)| = |G| = \aleph_0$ and $|C^*(G)| \geq 1$ and $|N^*(G)| \geq 1$.*

The proof of Theorem 3.9 will use the following lemmas:

Lemma 3.10. *If G is any graph and C is any finite end-cluster of G, then an odd number of nodes of C have odd degree in G.*

Proof. Let D denote the set of nodes which have odd degree in C and let n denote the node of C which is incident with a bridge of G. $|D|$ is even because C is finite. The set of nodes of C which have odd degree in G is $D - n$ if $n \in D$ and it is $D \cup \{n\}$ if $n \notin D$. In each case the number of nodes in it is odd.

Lemma 3.11. *If T is a tree, $\Delta(T)$ is infinite, and the number of $2-\infty$ ways in T is less than $|T|$, then T has $|T|$ end-nodes.*

Proof. Let δ denote the number of $2-\infty$ ways in T. Suppose first that T is locally finite. Then $|T| = \aleph_0$ by Theorem 2.5. There are nodes having arbitrarily high finite degrees in T because $\Delta(T) = \aleph_0$. If n is any node such that $d_T(n) \geqslant 2\delta + 2$, then there are at least $d_T(n) - 2\delta - 1$ end-nodes in T. For put $d_T(n) - 2\delta = \kappa$, then $\kappa \geqslant 2$ and there are at least κ edges incident with n which do not belong to any $2-\infty$ way, denote these edges by $e_1, ..., e_\kappa$. For $i = 1, ..., \kappa$, let T_i denote that connected component of $T - e_i$ to which n does not belong. $T_1, ..., T_\kappa$ are mutually disjoint trees and at most one of them contains a $1-\infty$ way. Hence, as may easily be seen, at least $\kappa-1$ of $T_1, ..., T_\kappa$ contain an end-node of T. So T contains at least $d_T(n) - 2\delta - 1$ end-nodes. It follows that the number of end-nodes in T is $\aleph_0 = |T|$ because T contains nodes of arbitrarily high finite degree.

Suppose next that T contains a node having infinite degree. Because $\delta < |T|$ there is at least one node whose degree is infinite and $> \delta$ by Theorem 2.7(B). If n is any such node, then the number of end-nodes of T is $\geqslant d_T(n)$ by an argument similar to the above. Therefore the number of end-nodes of T is $\geqslant \Delta(T)$. By Theorem 2.7(B), $\Delta(T) = |T|$. Lemma 3.11 is now proved.

Lemma 3.12. *Let G be a connected graph and G^* its cluster-bridge tree.*

(1) *G is the union of all its clusters and bridges.*

(2) $|C(G)| = |N(G^*)|$, $|E_{br}(G)| = |E(G^*)|$, $|E(G^*)| = |N(G^*)| - 1$, *where $C(G)$ denotes the set of all clusters of G.*

(3) *If G^* is infinite, then $|E(G^*)| = |N(G^*)| = |G^*|$.*

(4) *If G^* is infinite and $|C| \leqslant |G^*|$ for each cluster C of G, then $|G^*| = |G|$.*

(5) *Any cluster C of G is an end-cluster of G if and only if the node of G^* which corresponds to C is an end-node of G^*.*

(6) *If $\Delta(G^*)$ is infinite and G^* contains fewer than $|G^*|$ $2-\infty$ ways, then the number of end-clusters of G is $|G^*|$.*

Proof. (1), (2) and (3) follow easily from the definition of G.

To prove (4) observe that by (1),

$$G = E_{\mathrm{br}}(G) \cup \bigcup_{C \in C(G)} C.$$

By (2), $|E_{\mathrm{br}}(G)| = |E(G^*)|$ and $|C(G)| = |N(G^*)|$. By hypothesis, $|C| \leqslant |G^*|$ for each $C \in C(G)$. Therefore, $|G| \leqslant |E(G^*)| + |N(G^*)| \cdot |G^*|$. It follows that $|G| \leqslant |G^*|$ since G^* is infinite. On the other hand $|G| \geqslant |G^*|$ by (1) and (2). (4) is now proved.

(5) follows at once from the definitions, and (6) follows easily from (5) and Lemma 3.11. Lemma 3.12 is now proved.

Proof of Theorem 3.9. All the clusters of G are finite by Theorem 3.5. Hence G^* is infinite as G is. So by Lemma 3.12(4), $|G| = |G^*|$.

Suppose that at least one node has infinite degree in G, and let n denote any such node. Since G has no $2-\infty$ trail, we have by Theorem 2.8 that $E_{\mathrm{bl}}(n, G)$ is finite. It follows that $|E_{\mathrm{br}}(n, G)| = d_G(n)$. It follows from the definition of G^* that the node of G^* which corresponds to the cluster of G containing n has infinite degree in G^*, consequently $\Delta(G^*)$ is infinite. By Theorem 3.5, G^* contains no $2-\infty$ way. Hence by Lemma 3.12(6), the number of end-clusters of G is $|G^*|$. It has been shown that $|G^*| = |G|$. It now follows from Lemma 3.10 that $|N^*(G)| = |G|$.

By hypothesis, at least one node has infinite degree in G, and it was shown above that if n is any such node, then $E_{\mathrm{br}}(n, G) = d_G(n)$. Hence $|E_{\mathrm{br}}(G)| \geqslant d_G(n)$ for every node n having infinite degree in G. Therefore $|E_{\mathrm{br}}(G)| \geqslant \Delta(G)$. From this, $\Delta(G) \geqslant \aleph_0$ and from Theorem 2.7(B) it follows that $|E_{\mathrm{br}}(G)| = |G|$.

Consequently, if at least one node has infinite degree in G, then

$$|E_{\mathrm{br}}(G)| = |C^*(G)| = |N^*(G)| = \Delta(G) = |G|.$$

The alternative is that G is locally finite. In this case, because each cluster of G is finite, G^* is infinite and locally finite. By Theorem 2.1, therefore, G and G^* contain a $1-\infty$ way. G contains no $2-\infty$ way because it has no $2-\infty$ trail, G^* contains no $2-\infty$ way by Theorem 3.5.

Since G is connected and locally finite, by Theorem 2.5, $|G^*| = |G| = \aleph_0$.

Obviously, any tree with at least one edge and no $2-\infty$ way has at least one end-node. Consequently, by Lemma 3.12(5), $C^*(G) \neq \emptyset$, therefore, by Lemma 3.10, $N^*(G) \neq \emptyset$. Theorem 3.9 is now proved.

Corollary 3.13. *If G is an infinite connected graph and has no $1-\infty$ trail, then all the clusters of G are finite, at least one node has infinite degree in G and at least one node has infinite degree in G^*, and*

$$|E_{br}(G)| = |C^*(G)| = |N^*(G)| = \Delta(G) = |G|.$$

Remark 3.14. This corollary can of course be proved more simply without using Theorem 3.9.

If G is an infinite connected graph and does not have a $2-\infty$ trail and is locally finite, then in the notation of Theorem 3.9,

$$|N^*(G)| \geqslant |C^*(G)| = 1 + \sum_{n \in D_{\geqslant 3}(G)} (d_G(n) - 2)$$

as may easily be verified. $|C^*(G)| = 1$ if and only if G^* is a $1-\infty$ way.

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